ratio of phases; σ_I, abscissa of the point of inflection of the Buckley function; V, gas volume injected into the bed; T, duration of pumping; T_{0i}, duration of i-th gas withdrawal; t, time; x, spatial variable; τ, dimensionless time; ξ, dimensionless spatial coordinate.

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TWO TYPES OF HEAT TRANSFER IN MEDIA WITH THERMAL MEMORY

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It is shown that media with thermal memory can be grouped into two classes, based on different types of heat transfer. In media of the first class, the heat propagation velocity is infinite, while in the second class, it is finite. This difference is responsible for the peculiarities of the solutions of the heat-conduction problem in the two classes.

Currently in the study of heat- and mass-transfer processes under extreme conditions (low or very high temperatures), the mathematical formulation of heat conduction and mass exchange is used including differential memory of the medium [1-4, 7-9]. A linerarized heat-conduction equation of this kind was first obtained in [8]; it describes heat transfer with a finite heat propagation velocity [8, 9]. In the derivation of a similar heat-conduction equation in [2], a different, more general form of the linearized integral heat-transfer relation was used, which includes the instantaneous values $\lambda(0)$ and $c(0)$ of the relaxation functions for the heat flux and the internal energy. Then media with transient thermal memory can naturally be divided into two classes: those with the instantaneous value $\lambda(0) > 0$ (Fourier media) and those with $\lambda(0) = 0$ (Maxwellian media). It was also shown in [2] that the Nunziato heat-conduction equation with $\lambda(0) = 0$ can be reduced to the Pipkin–Curtin equation [8] and hence in this type of medium, heat propagates with a finite velocity. It is shown below that in a Fourier medium, heat is transferred with an infinite velocity. Using the method of solving the heat-conduction problem for the Nunziato equation worked out in [4], we describe the heat-conduction behavior for small values of the time in both types of media. The results are applied to the distribution function of an instantaneous point source and this allows one to deduce the type of heat propagation and also the qualitative features of the solution for each type of medium.

We consider the integrodifferential heat-conduction equation for the function $u(t, M) = T(t, M) - T(0, M)$ describing the linearized transfer process with transient thermal memory as formulated by Nunziato [2]:

$$\frac{c_1(0)}{a_0}\frac{\partial u}{\partial t} - \lambda_1(0)\Delta u + \int_0^\infty \left[\frac{1}{a_0}\frac{dc_1(\tau)}{d\tau}\frac{\partial u(t-\tau, M)}{\partial t} - \frac{d\lambda_1(\tau)}{d\tau}\Delta u(t-\tau, M)\right] d\tau = \frac{b(t, M)}{\lambda_0}, \quad (1)$$

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where $u(0, M) = 0$, $\lambda_0 = \lambda(\infty)$; $\rho_0 c_0 = c(\infty)$. In transform space this equation has the form

$$\Delta U(p, M) - \frac{\varphi(p)}{a_0} U(p, M) = \frac{B(p, M)}{\lambda_0 p \Lambda_1(p)}; \varphi(p) = pC_1(p)/\Lambda_1(p). \tag{2}$$

Where as usual, capital letters denote a Laplace transform of the corresponding function. From the definitions of λ_0, ρ_0, and c_0, the relaxation kernels $\lambda_1(t)$ and $c_1(t)$ approach unity for large times. According to a theorem on limits [5], the Laplace transforms have the following behavior as $p \to 0$: $\Lambda_1(p) \to p^{-1}$; $C_1(p) \to p^{-1}$ $\varphi(p) \to p$, and (2) for the transform function $U(p, M)$ takes the usual form corresponding to Fourier heat transfer. Thus, the solutions for the different heat-conduction equations (the usual Fourier equation and the equation with transient memory) should asymptotically approach each other as the time increases, and converge to some stationary state. From (1) and (2) it can be seen that if the time dependence of the heat flux matches that of the internal energy, i.e., $\lambda_1(t) = c_1(t)$ and $c_1(0) > 0$, then the heat-conduction equation remains parabolic but with a change in the heat source. According to the formulation in [4], the transform heat-conduction problem without boundary conditions for the homogeneous forms of (1) is obtained by replacing p by $\varphi(p)$ in the transform solution $U_1(p, M)$ for the ordinary heat-conduction equation. The solution $u(t, M)$ is obtained using the following result derived in [4]:

$$u(t, M) = \int_0^\infty a(t, \tau) u_1(\tau, M) d\tau, \quad a(t, \tau) = L^{-1} \exp[-\varphi(p)\tau], \tag{3}$$

where $u_1(t, M) = L^{-1}U_1(p, M)$. The result (3) is correct when the function $\varphi(p)$ satisfies the condition $\mathrm{Re}[\varphi(p)] > \sigma$ for $\mathrm{Re}(p) > \sigma$, which is called the condition of "transformability" in [4]. Note that this condition is satisfied if the function $\varphi(p)$ approaches $\beta_0 p^\alpha$ as $p \to \infty$, where $\beta_0 > 0$ and $-1 \le \alpha \le 1$.

For small times, when the solution of the heat-conduction equation with memory differs most from that of ordinary heat conduction, it is useful to expand the relaxation kernels in powers of t:

$$\begin{aligned} \lambda_1(t) &= \lambda_1(0) + \sum_{k=1}^{\infty} \frac{\lambda^{(k)}}{k!} t^k; \quad \lambda^{(k)} = \frac{d^k \lambda_1(0)}{dt^k}; \quad \lambda_1(0) \geqslant 0; \\ c_1(t) &= c_1(0) + \sum_{k=1}^{\infty} \frac{c^{(k)}}{k!} t^k; \quad c^{(k)} = \frac{d^k c_1(0)}{dt^k}; \quad c_1(0) > 0. \end{aligned} \tag{4}$$

At large p (corresponding to small times) the Laplace transforms $\Lambda_1(p)$ and $C_1(p)$ of these kernels have the form

$$\Lambda_1(p) = \frac{\lambda_1(0)}{p} + \sum_{k=1}^{\infty} \frac{\lambda^{(k)}}{p^{k+1}}; \quad C_1(p) = \frac{c_1(0)}{p} + \sum_{k=1}^{\infty} \frac{c^{(k)}}{p^{k+1}}. \tag{4a}$$

Depending on the small time behavior of the relaxation kernels $\lambda_1(t)$ and $c_1(t)$, the expansion of $\varphi(p)$ can be divided naturally into two types, and the behavior of the solution of the heat-conduction problem with memory critically depends on the type of $\varphi(p)$ expansion.

Relaxation Kernel of the First Class $\lambda_1(0) > 0$. In this case the expansion of the function $\varphi(p)$ for large p in decreasing powers of p begins with a linear term and satisfies the transformability condition

$$\begin{aligned} \varphi(p) &= d_1 p + d_0 + \sum_{m=1}^{\infty} d_{-m} p^{-m}; \quad d_1 > 0; \\ d_1 &= c_1(0)/\lambda_1(0); \quad d_0 = d_1 \left[\frac{c^{(1)}}{c_1(0)} - \frac{\lambda^{(1)}}{\lambda_1(0)}\right]; \\ d_{-1} &= d_1 \left[\frac{c^{(2)}}{c_1(0)} - \frac{\lambda^{(2)}}{\lambda_1(0)}\right] - d_0 \frac{\lambda^{(1)}}{\lambda_1(0)}; \quad \ldots \end{aligned} \tag{5}$$

In the analysis of the solution $u(t, M)$ of the heat-conduction equation (1) for small times, one can keep the first two terms in expansion (5), i.e., put $\varphi(p) = d_1 p + d_0$. Then (3) for $u(t, M)$ leads as usual to a change in the time scale and to the appearance of an exponential factor when compared to the solution for the ordinary heat-conduction equation.

$$u(t, M) = L^{-1} U_1(d_1 p + d_0, M) = \frac{1}{d_1} u_1\left(\frac{t}{d_1}, M\right) \exp\left(-\frac{d_0 t}{d_1}\right). \tag{6}$$

The small time behavior of the solution can be made more exact by including higher-order terms in expansion (5) for $\varphi(p)$. For example, if we keep the third-order term in (5) and use the results of [4] and the properties of the Laplace transform [5, 6], we find

$$u(t, M) = L^{-1}U_1(d_1p + d_0 + d_{-1}p^{-1}, M) = u^{[1]}(t, M) - \operatorname{sgn}(d_{-1}) \int_0^t u^{[1]}(\tau, M) \sqrt{\frac{|d_{-1}|\tau}{(t-\tau)}} L_1[2\sqrt{|d_{-1}|\tau(t-\tau)}]\, d\tau. \quad (6a)$$

where

$$\operatorname{sgn}(d_{-1}) = \begin{cases} 1, & d_{-1} > 0; \\ 0, & d_{-1} = 0; \\ -1, & d_{-1} < 0; \end{cases} \quad L_1 = \begin{cases} J_1, & d_{-1} > 0; \\ I_1, & d_{-1} < 0, \end{cases}$$

and $u^{[1]}(t, M)$ denotes the solution $u(t, M)$ obtained in (6).

We apply (3) to the distribution function for an instantaneous point source placed at point M_0 of an n-dimensional space at $t = 0$. Since [7]

$$f_1(t, r) = \frac{\exp(-r^2/4a_0t)}{(2\sqrt{\pi a_0 t})^n}; \; r = |\overrightarrow{MM_0}| = \left[\sum_{i=1}^{n} (x_i - x_{i0})^2\right]^{1/2},$$

then

$$f(t, r) = \int_0^\infty \frac{a(t, \tau)}{(2\sqrt{\pi a_0 t})^n} \exp\left(-\frac{r^2}{4a_0\tau}\right) d\tau; \; n = 1, 2, 3,$$

where r is the distance between points M and M_0. With the help of (6) and (6a), the small time behavior of the distribution function f(t, r) of an instantaneous point source can be worked out to first and second order

$$f^{[1]}(t, r) = \frac{1}{d_1}\left(\frac{\sqrt{d_1}}{2\sqrt{\pi a_0 t}}\right)^n \exp\left(-\frac{d_0}{d_1}t - \frac{r^2 d_1}{4a_0 t}\right); \quad (7)$$

$$f(t, r) = f^{[1]}(t, r) - \operatorname{sgn}(d_{-1}) \int_0^t f^{[1]}(\tau, r) \sqrt{\frac{|d_{-1}|\tau}{(t-\tau)}} L_1[2\sqrt{|d_{-1}|\tau(t-\tau)}]\, d\tau. \quad (7a)$$

The first-order equation (7) shows that for relaxation kernels of the first class, heat propagates with an infinite velocity, as in the case of Fourier's law. If one includes the third-order term in the expansion of $\varphi(p)$ (see (7a)), the solution becomes quantitatively more accurate, but the qualitative nature of the heat propagation is not changed. It is obvious that the higher-order terms of expansion (5) do not change the nature of the heat propagation. At large times the distribution function for the instantaneous point source approaches its classical form. We will refer to a medium with a relaxation kernel of the first kind as a Fourier medium. The solution of the heat-conduction problem for a medium with thermal memory of the Fourier type has the same qualitative features as in ordinary heat conduction. This can be seen, for example, in the expressions for the velocity $w(\omega) = w_{cl}(\omega)/\sqrt{d_1}$ and the damping coefficient $\xi(\omega) = \xi_{cl}(\omega)\sqrt{d_1}$ in the limit $\omega \to \infty$ obtained in the temperature wave problem in [2]. This also explains the infinite initial heat flux at the boundary ($x = 0$) for a unit step temperature input at the boundary of a semiinfinite medium, as will now be shown. The transform solution of this problem has the form

$$U(p, x) = \frac{1}{p} \exp\left[-\frac{x}{\sqrt{a_0}}\sqrt{\varphi(p)}\right], \; x \geqslant 0.$$

Using the two-term approximation to (5) for $\varphi(p)$ as a function of p, and using the table of transforms in [6] and also Eq. (6), we obtain the solution at small times

$$u(t, x) = \frac{1}{2}\exp\left(\sqrt{\frac{d_0}{a_0}}x\right)\operatorname{erfc}\left(\frac{\sqrt{d_1}x}{2\sqrt{a_0 t}} + \sqrt{\frac{d_0 t}{d_1}}\right) + \frac{1}{2}\exp\left(-\sqrt{\frac{d_0}{a_0}}x\right)\operatorname{erfc}\left(\frac{\sqrt{d_1}x}{2\sqrt{a_0 t}} - \sqrt{\frac{d_0 t}{d_1}}\right). \quad (8)$$

Differentiating (8) and using (4) and the relation for the heat flux in [2], we obtain the flux at the boundary $q_0(t)$:

$$q_1 = -\lambda_0 \frac{\partial u(t, 0)}{\partial x} = \sqrt{\lambda_0\rho_0 c_0}\left[d_0 \operatorname{erfc}\left(\sqrt{\frac{d_0 t}{d_1}}\right) + \sqrt{\frac{d_1}{\pi t}}\exp\left(-\frac{d_0 t}{d_1}\right)\right];$$

$$q_0(t) = \lambda_1(0)\, q_1(t) + \int_0^t q_1(t-\tau)\, [\lambda^{(1)} + \lambda^{(2)}\tau + o(\tau)]\, d\tau,$$

from which we see that $q_0(t) \to \infty$ as $t \to 0$, as expected

$$q_0(t) \to \lambda_1(0) \sqrt{\frac{\lambda_0 \rho_0 c_0 d_1}{\pi t}} + O(1) = \lambda_1(0) \sqrt{d_1} (q_0)_{cl} + O(1).$$

Here $(q_0)_{cl}$ is the classical value of the heat flux at the boundary for the same problem [7]. One can refine the solution u(t, x) with the help of (6a).

Relaxation Kernel of the Second Class. $\lambda_1(0) = 0$, $\lambda^{(1)} > 0$. The expansion of $\varphi(p)$ for large p (corresponding to small times) starts with p^2:

$$\varphi(p) = d_2 p^2 + d_1 p + d_0 + \sum_{k=1}^{\infty} d_{-k} p^{-k}, \quad d_2 > 0; \tag{9}$$

$$d_2 = \frac{c_1(0)}{\lambda^{(1)}}; \quad d_1 = d_2 \left[\frac{c^{(1)}}{c_1(0)} - \frac{\lambda^{(2)}}{\lambda^{(1)}} \right];$$

$$d_0 = d_2 \left[\frac{c^{(2)}}{c_1(0)} - \frac{\lambda^{(3)}}{\lambda^{(1)}} \right] - d_1 \frac{\lambda^{(2)}}{\lambda^{(1)}}; \ldots .$$

Unlike (5), expansion (9) does not satisfy the transformability condition. Therefore, (3) cannot be used directly for the solution of the heat-conduction problem (1) with or without memory. However, if we use the fact that the solution of the heat-conduction equation (1) in transform space does not contain $\varphi(p)$ itself, but rather $\varphi_0(p) = \sqrt{\varphi(p)}$, i.e., $U(p, M) = U_1[\varphi(p), M] = U_2[\varphi_0(p), M]$, where $\varphi_0(p)$ satisfies the transformability condition, then (3) can be applied to the transform $U_2[\varphi_0(p), M]$.

With the help of the above discussion, we analyze the behavior of the solution u(t, M) at small times. Keeping terms up to third order in the expansion (9) and using (3), we transform to the solution u(t, M) using the tables in [5, 6] and straightforward algebraic manipulation:

$$a(t, \tau) = L^{-1} \exp[[-\tau \sqrt{d_2 p^2 + d_1 p + d_0}] = \exp\left(-\frac{d_1}{2d_2} t\right) \left[\delta(t - \sqrt{d_2}\tau) + b\sqrt{d_2}\tau E(t - \sqrt{d_2}\tau) \frac{I_1(b\sqrt{t^2 - d_2\tau^2})}{\sqrt{t^2 - d_2\tau^2}} \right], \tag{10}$$

$$b \equiv \frac{d_1}{2d_2} \sqrt{1 - \frac{4d_0 d_2}{d_1^2}}.$$

Therefore

$$u(t, M) = \int_0^\infty a(t, \tau) u_2(\tau, M) d\tau = \exp\left(-\frac{d_1}{2d_2} t\right) \left[\frac{u_2\left(\frac{t}{\sqrt{d_2}}, M\right)}{\sqrt{d_2}} + \frac{b}{\sqrt{d_2}} \int_0^t u_2\left(\frac{\tau}{\sqrt{d_2}}, M\right) \frac{\tau I_1(b\sqrt{t^2 - \tau^2})}{\sqrt{t^2 - \tau^2}} d\tau, \right. \tag{11}$$

where $u_2(\mu, M) = L^{-1}U_1(p^2, M)$ and $U_1(p, M)$ is the transform solution of the same problem for the ordinary heat-conduction equation. For a Maxwell medium the relation between the solutions u(t, M) and $u_1(t, M)$ for the same heat-conduction problem is more complicated than for a thermal medium of the Fourier type.

The transform $F_2(p, r)$ of the distribution function $f_1(t, r)$ for an instantaneous point source is written in the form

$$F_2(p, r) = F_1(p^2, r) = \begin{cases} \dfrac{\exp(-|x - x_0| p / \sqrt{a_0})}{2\sqrt{a_0}\, p}, & n = 1; \\[2ex] \dfrac{\exp(-rp/\sqrt{a_0})}{4\pi a_0 r}, & n = 3; \end{cases}$$

Applying (3) to this expression we find f(t, r) for the instantaneous point source:

$$f_2(t,\ r) = L^{-1}F_2(p,\ r) = \begin{cases} \dfrac{1}{2\sqrt{a_0}} E\left(t - \dfrac{|x - x_0|}{\sqrt{a_0}}\right), & n = 1; \\ \dfrac{1}{4\pi a_0 r}\,\delta\left(t - \dfrac{r}{\sqrt{a_0}}\right), & n = 3; \end{cases}$$

$$f(t,\ r) = \begin{cases} \dfrac{1}{2\sqrt{a_0}} \displaystyle\int_{\frac{|x-x_0|}{\sqrt{a_0}}} a(t,\ \tau)\,d\tau, & n = 1; \\ \dfrac{1}{4\pi a_0 r}\, a\left(t,\ \dfrac{r}{\sqrt{a_0}}\right), & n = 3. \end{cases}$$

For small times, the approximation formulas (10), (11) give the distribution function of the instantaneous point source for n = 1:

$$f(t,\ x) = \frac{1}{2\sqrt{a_0 d_2}} \exp\left(-\frac{d_1}{2d_2}\,t\right)\left[E\left(\frac{t}{\sqrt{d_2}} - \frac{|x - x_0|}{\sqrt{a_0}}\right) + b\int_0^t E\left(\frac{\tau}{\sqrt{d_2}} - \frac{|x - x_0|}{\sqrt{a_0}}\right)\frac{\tau I_1\left(b\sqrt{t^2 - \tau^2}\right)}{\sqrt{t^2 - \tau^2}}\,d\tau\right] =$$

$$= \frac{\gamma}{w} I_0\left(b\sqrt{t^2 - \frac{|x - x_0|^2}{w^2}}\right) E\left(t - \frac{|x - x_0|}{w}\right)\exp(-d_1\gamma t); \tag{12}$$

$$\gamma = \frac{w^2}{2a_0};\ b = d_1\gamma\sqrt{1 - \frac{2d_0}{\gamma d_1^2}},$$

where $w = \sqrt{a_0/d_2}$ is the propagation velocity of the thermal wave front. For n = 3 the function f(t, r), after simple transformations, takes the form

$$f(t,\ r) = \exp(-d_1\gamma t)\left[\frac{1}{4\pi a_0 r}\,\delta\left(t - \frac{r}{w}\right) + \frac{b}{4\pi a_0 w} E\left(t - \frac{r}{w}\right)\frac{I_1\left(b\sqrt{t^2 - \frac{r^2}{w^2}}\right)}{\sqrt{t^2 - \frac{r^2}{w^2}}}\right]. \tag{13}$$

In the special case ($d_1 = 1$, $d_0 = 0$) these expressions reduce to the fundamental solution for the hyperbolic heat-conduction equation. Results (12) and (13) show that in one dimension and three dimensions, heat propagates with a finite velocity, for relaxation kernels of the second class, as in the case of the Maxwell law. It is evident that inclusion of the higher-order terms in (9) will not change the qualitative nature of the heat propagation. We will refer to a medium with a relaxation kernel of the second class, and a Maxwellian medium. The finite heat-propagation velocity is consistent with the damping coefficient $\xi(\omega) \to d_1\gamma$ and propagation velocity $w(\omega) \to w = \sqrt{a_0/d_2}$ in the limit $\omega \to \infty$ obtained in the temperature wave problem of [2]. In heat-conduction problems in media with memory of the Maxwellian type, other qualitative features of the solution can also be predicted: the presence of a sharply defined leading edge to the thermal wave with a constant translational velocity and a spreading diffusive wake behind it, the propagation of damped thermal shock waves, a limit to the maximum heat flux at the boundary for a unit temperature step input there, etc; and these effects being consequences of the finite heat propagation velocity. In general, the heat-conduction problem for a Maxwellian medium shares the qualitative features of the solution for the same problem with the hyperbolic heat-conduction equation.

We discuss the behavior of the function f(t, r) at small times. Equations (12) and (13) are written under the assumption that $d_1^2 > 4d_2d_0$ (or $b > 0$). Therefore at fixed t, the one-dimensional instantaneous point source (12) leaves an aperiodic spatial wake that is constant in sign. The maximum value f_{max} occurs at $x = 0$ ($f_{max} = (\gamma/w)I_0(bt)\exp(-d_1\gamma t)$ and it decreases monotonically with increasing x to the value f_{min} ($f_{min} = (\gamma/w)\exp(-d_1\gamma t)$, at the wave front. If $d_1^2 = 4d_2d_0$ (or $b = 0$), the one-dimensional instantaneous point source leaves a spatial wake of the form

$$f(t,\ x - x_0) = \frac{\gamma}{w} E\left(t - \frac{|x - x_0|}{w}\right)\exp(-d_1\gamma t);\ b = 0. \tag{12a}$$

When $d_1^2 < 4d_2d_0$, then b is imaginary ($b = ib_0$, where $b_0 = \gamma\sqrt{4d_2d_0 - d_1^2} > 0$). This leads to the Bessel function J_0 in (12):

$$f(t,\ x-x_0)=\frac{\gamma}{w}J_0\left(b_0\sqrt{t^2-\frac{|x-x_0|^2}{w^2}}\exp(-d_1\gamma t)\,\mathrm{E}\left(t-\frac{|x-x_0|}{w}\right)\right). \tag{12b}$$

From this we see that for fixed t, the function $f(t,\ x-x_0)$ oscillates in space and is damped with the amount of damping increasing with decreasing x. The maximum value occurs on the wave front $f_{max}=(\gamma/w)\exp(-d_1\gamma t)$. These three types of spatial forms also occur in the three-dimensional case for the diffusive part of the instantaneous point source (see (13)):

$$f(t,\ r)=\frac{\exp(-d_1\gamma t)}{4\pi a_0 r}\,\delta\left(t-\frac{r}{w}\right);\ d_1^2=4d_0d_2; \tag{13a}$$

$$f(t,\ r)=\exp(-d_1\gamma t)\left[\frac{1}{4\pi a_0 r}\,\delta\left(t-\frac{r}{w}\right)-\frac{b_0}{4\pi a_0 w}\,\mathrm{E}\left(t-\frac{r}{w}\right)\frac{J_1\left(b_0\sqrt{t^2-\frac{r^2}{w^2}}\right)}{\sqrt{t^2-\frac{r^2}{w^2}}}\right];\ d_1^2<4d_2d_0. \tag{13b}$$

We study the small-time behavior of the heat-conduction boundary-value problem with a unit step temperature input at the boundary of a semiinfinite medium. The transform solution of this problem for the heat-conduction equation (1) is given by

$$U(p,\ x)=\frac{1}{p}\exp\left[-\frac{x}{\sqrt{a_0}}\varphi_0(p)\right];\ \varphi_0(p)=\sqrt{d_2p^2+d_1p+d_0}. \tag{14}$$

Where we have used the three-term approximation for $\varphi(p)$ in the expansion (9). Using (10) and the properties of the inverse Laplace transform, the solution u(t, x) for small times is obtained as

$$u(t,\ x)=E\left(t-\frac{x}{w}\right)\left[\exp\left(-\frac{d_1\gamma x}{w}\right)+\frac{bx}{w}\int\limits_{\frac{x}{w}}^{t}\frac{I_1\left(b\sqrt{\tau^2-\frac{x^2}{w^2}}\right)}{\sqrt{\tau^2-\frac{x^2}{w^2}}}\exp(-d_1\gamma\tau)\,d\tau.\right. \tag{15}$$

The result (15) shows that a damped thermal shock wave propagates, and the temperature jump u_0 at the wave front is given by $u_0=\exp(-d_1\gamma x/w)$. This expression for the temperature jump differs from that of [10] for the hyperbolic heat-conduction equation, only by the factor d_1 in the exponent. Solving this equation for x, we obtain formulas for the penetration depth and penetration time for thermal waves in the medium as a function of the value of the temperature jump at the wave front:

$$x\leqslant 2\frac{d_2}{d_1}w\,|\ln u_0|;\ t\leqslant 2\frac{d_2}{d_1}|\ln u_0|;\ u_0\sim 10^{-3}-10^{-4}.$$

These quantities can be used to define approximately the region of nonparabolicity of the problem. Differentiating (15) and using the relation for the heat flux given in [2], we find a simple expression for $q_0(t)$ at small times:

$$q_0(t)=\frac{\lambda_0 w}{a_0}c_1(0)\left[1+\left(\frac{\lambda^{(2)}}{\lambda^{(1)}}+d_1\gamma\right)t+\left(d_1\gamma\frac{\lambda^{(2)}}{\lambda^{(1)}}+\frac{1}{2}\frac{\lambda^{(3)}}{\lambda^{(1)}}+\frac{\gamma^2d_1^2}{2}-\frac{d_0}{2}\right)t^2+o(t^2)\right]. \tag{16}$$

From this result we see that the flux at the boundary is finite at t = 0; $q_0(0)=c_1(0)\lambda_0 w/a_0$ and differs from the value in [7] for the hyperbolic heat-conduction equation only by the factor $c_1(0)$. Hence, the qualitative features of heat conduction in a Maxwellian-type medium at small times are the same as those for the hyperbolic heat-conduction equation; the expansion parameters of (9) for the relaxation kernel show up in the quantitative relations. From the expression for the temperature jump u_0, we see that the attenuation of propagating waves can be greater or lesser than in the hyperbolic case, depending on the form of the relaxation kernels. The heat-conduction problem in a semiinfinite medium with an arbitrary temperature distribution on the boundary was solved in [9] using the Pipkin–Curtin equation and a more approximate method using combined Fourier and Laplace transforms.

Finally, we note that recent studies of mass-transfer processes have also used linear integrodifferential equations of type (1) but with a single relaxation function [3], since there is not as yet enough experimental justification for the introduction of a relaxation function for the internal energy. The results given above also apply for diffusion processes, and one can, in similar fashion, define a Fick's type medium with an infinite mass propagation velocity and a Maxwellian medium with a finite mass propagation velocity. The results obtained here are valid for mass-transfer processes in these materials.

NOTATION

λ_0, equilibrium thermal conductivity; ρ_0, mass density, c_0, equilibrium heat capacity of the material; $\lambda(t)$, $c(t)$, relaxation kernels for the heat flux and internal energy; $a_0 = \lambda_0/\rho_0 c_0$, thermal diffusivity, T, temperature; L, L^{-1}, Laplace transform and inverse Laplace transform operators; p, Laplace transform variable; $\lambda_1(t) = \lambda(t)/\lambda_0$, $c_1(t) = c(t)/c_0\rho_0$, dimensionless relaxation kernels for the heat flux and internal energy; M, spatial point; J_1, I_1, first-order Bessel functions of the first kind for real and imaginary argument; J_0, I_0, zeroth-order Bessel functions of the first kind for real and imaginary argument; w, heat propagation velocity; i, imaginary unit; Re, real part of a complex number or function; σ, index of increase of the function u_1; $\delta(t)$, the Dirac delta function; E(t), Heaviside unit step function.

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TEMPERATURE DISTRIBUTION IN PLATES AND INFINITE PRISMATIC BODIES OF COMPLEX CROSS SECTION FOR A TIME-VARYING HEAT-TRANSFER COEFFICIENT

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We present a new method for solving heat-conduction problems with a time-varying heat-transfer coefficient in domains of complex shape, and cite numerical results for two problems.

Because of mathematical difficulties, heat-conduction problems with a time-dependent heat-transfer coefficient cannot be solved analytically in complex domains for a given Bi(Fo), even for one-dimensional cases [1].

We consider the case when the calculation of the temperature distribution in plates and infinite prismatic bodies of complex cross section reduces to the solution of the heat-conduction problem

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